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QUANTUM DEFORMATIONS OF THE SELF-DUALITY EQUATION AND CONFORMAL TWISTORS

B.M.Zupnik
Bogoliubov Laboratory of Theoretical Physics , JINR, Dubna, Moscow Region, 141980, Russia, E-mail: zupnik@thsun1.jinr.dubna.su

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Abstract. A noncommutative algebra of the complex q-twistors and their differentials is considered on the basis of the quantum $GL_q(4) \times SL_q(2)$ group. Real and pseudoreal q-twistors are discussed too. We consider the quantum-group self-duality equation in the framework of the local gauge algebra of differential forms on q-twistor spaces. Quantum deformations of the general multi-instanton solutions are constructed. The corresponding noncommutative algebras of moduli are introduced. The general q-instanton connection is a function of the q-twistors and the q-moduli .

1 Introduction

Noncommutative generalizations of the local gauge theories have been considered in the framework of different approaches [1-7]. The mathematically strict approach of Refs[3,4] is based on the noncommutative global generalizations of the classical fibre bundles. We prefer to study the local structure of the quantum-group gauge theory in terms of the deformed connection and curvature differential forms A and F. The basic algebra of these gauge forms should be covariant under the action of the quantum gauge group [5,6]

$$A \rightarrow TAT^{-1} + dTT^{-1} \tag{1.1}$$

where T and dT are elements of the differential complex on the quantum group.

Consider some classical or quantum space with the coordinates z and let $T(z)$ and $dT(z,dz)$ be noncommutative 'functions' on this space. We shall treat these functions as generators of the local gauge differential complex if the map

$$T \rightarrow T(z), \qquad dT \rightarrow dT(z,dz) \tag{1.2}$$

conserves all relations between T and dT.

The general function $T(z)$ is some formal expansion with noncommutative coefficients. Thus, the localization of the quantum group is equivalent to the definition of an infinite-dimensional noncommutative Hopf algebra.

The q-deformations of the Grassmann and twistor spaces were studied in Refs[8-10]. In section 2 we consider the differential calculus on the 4-dimensional deformed complex twistor space $T_q(4,C)$. The real forms of the q-twistor space are also discussed .

Section 3 is devoted to the description of the quantum-group gauge fields on the q-twistor space. We consider the algebraic relation for the connection form in the $GL_q(N)$ gauge theory [5,6] that defines the algebraic properties of the 'off-shell' gauge fields. The generalizations of the Yang-Mills and self-duality equations are discussed. Note that one can use the 'pure gauge' $U(1)$ field and q-traceless curvature 2-forms in the $U_q(N)$ gauge theory [7]. The noncommutative analogue of the BPST one-instanton solution [11] was constructed in the deformed 4-dimensional Euclidean space [7].

The deformed analogue of the t'Hooft multi-instanton twistor solution for the gauge group $GL_q(2)$ (or $U_q(2)$) is considered in section 4. We use a multidimensional extension of the q-twistor algebra by the set of noncommutative $6D$-vector generators b. The potential of our solution is a sum of the central (z,b)-functions obeying the q-twistor Laplace equation.

The deformed generalization of the Atiyah-Drinfel'd-Hitchin-Manin solution [12] for the gauge group $GL_q(N)$ contains q-twistor functions u and $\tilde{u}$. We generalize the classical conformal constructions of Ref[13]. One can consider the linear twistor functions v and $\tilde{v}$ that depend on the noncommutative moduli b and $\tilde{b}$. The functions u and v are submatrices of the quantum $GL_q(N+2p)$ matrix for the instanton number p. The consistency relations for the deformed ADHM-construction can be proved in the framework of a differential calculus on $GL_q(N+2p)$. The self-duality condition is equivalent to the bilinear constraint on the moduli b and $\tilde{b}$.

A preliminary version of this work was published in Ref[14]. Note that we use here the modified notation and definitions of some basic quantities.

2 Differential calculus on the deformed twistor space

The conformal covariant description of the classical ADHM solution was considered in Ref[13]. This approach uses real forms of the complex $GL(4,C) \times SL(2,C)$ twistors where $GL(4,C)$ is the complex conformal group. It is convenient to discuss firstly the deformed complex twistors.

Let R^{ab}_{cd} be a solution of the Yang-Baxter equation satisfying also the Hecke relation

$$R\,R'\,R = R'\,R\,R' \tag{2.1}$$

$$R^2 = I + (q - q^{-1})R \tag{2.2}$$

where q is a complex parameter. Note that the standard notation for these R-matrices is $R = \hat{R}_{12},\ R' = \hat{R}_{23}$ [15]. We use the symbols $a, b \ldots h = 1 \ldots 4$ for the 4D-spinor indices.

The multiparameter 4D R-matrix [16-18] and corresponding inverse matrix R^{-1} can be written in the following simple form:

$$(R^{\pm 1})^{ab}_{cd} = \delta^a_c \delta^b_d [q^{\pm 1} - q^{\epsilon(a-b)}] + r(ab)\delta^a_d \delta^b_c \tag{2.3}$$

where $\epsilon(b-a) = 0,\ \pm 1$ is a sign function and $r(ab)$ are complex parameters satisfying the relations $r(ab)r(ba) = 1$. It is evident that this formula is valid for an arbitrary number N .

The standard $GL_q(4)$ solution corresponds to the case $r(ab) = 1$ [15]. The choice $q = 1$ leads to the unitary R-matrix [19]

$$R^2 = I \tag{2.4}$$

Consider also the $SL_q(2,C)$ R-matrix

$$R^{\alpha\beta}_{\mu\nu} = q\delta^\alpha_\mu \delta^\beta_\nu + \varepsilon^{\alpha\beta}(q)\varepsilon_{\mu\nu}(q) \tag{2.5}$$

where $\varepsilon(q)$ is the deformed antisymmetric symbol

$$\varepsilon^{12}(q) = -\varepsilon_{12}(q) = \frac{1}{\sqrt{q}}, \quad \varepsilon^{21}(q) = -\varepsilon_{21}(q) = -\sqrt{q} \tag{2.6}$$

The q-deformed flag spaces and twistors were considered in Refs[8-10]. We shall treat the complex q-twistors z^α_a as generators of the noncommutative algebra with the basic relation

$$R^{\alpha\beta}_{\mu\nu}\, z^\mu_a\, z^\nu_b = z^\alpha_c\, z^\beta_d\, R^{dc}_{ba} \tag{2.7}$$

This relation for the (4×2) rectangular matrix z is analogous to the RTT-relations for the square quantum matrices. The consistency conditions for (2.7) are pairs of the Yang-Baxter and Hecke relations (2.1, 2.2) for the independent 4D and 2D R-matrices with the unique common parameter q.

The differential calculus on the complex q-twistor space $T_q(4,C)$ can be constructed by the analogy with the bicovariant differential complex on the quantum linear group [20-23]. Consider the relations between z^α_a and their differentials dz^α_a

$$z^\alpha_a \, dz^\beta_b = R^{\alpha\beta}_{\mu\nu} \, dz^\mu_c \, z^\nu_d \, R^{dc}_{ba} \tag{2.8}$$

$$dz^\alpha_a \, dz^\beta_b = -R^{\alpha\beta}_{\mu\nu} \, dz^\mu_c \, dz^\nu_d \, R^{dc}_{ba} \tag{2.9}$$

The elements z and dz are generators of the external algebra $\Lambda T_q(4,C)$ on the q-twistor space. The operator of external derivative d on $\Lambda T_q(4,C)$ is nilpotent and satisfies the ordinary Leibniz rule.

The symmetry properties of dz can be obtained from Eq(2.9)

$$P^{(+)}_2 dz \, dz' P^{(+)}_4 = 0 = P^{(-)}_2 dz \, dz' P^{(-)}_4 \tag{2.10}$$

where $P^{(\pm)}_2$ and $P^{(\pm)}_4$ are the projection operators for $SL_q(2)$ and $GL_q(4)$, respectively [15]

$$P^{(+)} + P^{(-)} = I, \quad R = qP^{(+)} - q^{-1}P^{(-)} \tag{2.11}$$

One can define the algebra of partial derivatives ∂^a_α on $T_q(4,C)$

$$R^{ab}_{cd} \, \partial^c_\alpha \, \partial^d_\beta = \partial^a_\mu \, \partial^b_\nu \, R^{\nu\mu}_{\beta\alpha} \tag{2.12}$$

$$\partial^a_\alpha \, z^\beta_b = \delta^a_b \, \delta^\beta_\alpha + R^{\beta\mu}_{\alpha\nu} \, R^{da}_{cb} \, z^\nu_d \, \partial^c_\mu \tag{2.13}$$

$$\partial^a_\alpha \, dz^\beta_b = R^{\beta\mu}_{\alpha\nu} \, R^{da}_{cb} \, dz^\nu_d \, \partial^c_\mu \tag{2.14}$$

Consider a definition of the deformed ε_q-symbol for $GL_q(4)$

$$\varepsilon^{abcd}_q = -qR^{ba}_{fe} \, \varepsilon^{efcd}_q = [P^{(-)}_4]^{ba}_{fe} \, \varepsilon^{efcd}_q = -q^{\epsilon(b-a)} r(ba) \, \varepsilon^{bacd}_q \tag{2.15}$$

Analogous relations are valid for other neighboring pairs of indices.

The q-twistors obey the following identity:

$$\varepsilon^{abcd}_q \, z^\beta_b \, z^\mu_c \, z^\nu_d = 0 \tag{2.16}$$

Introduce the $SL_q(2)$-invariant bilinear function of q-twistors

$$y_{ab} = \frac{q^2}{1+q^2} \, \varepsilon_{\alpha\beta}(q) \, z^\alpha_a \, z^\beta_b = [P^{(-)}_4]^{dc}_{ba} \, y_{cd} \tag{2.17}$$

This vector satisfies the following commutation relation

$$y_{ab} \, z^\alpha_c = q^{-1} R^{ed}_{ha} R^{fh}_{cb} \, z^\alpha_d \, y_{ef} \tag{2.18}$$

In consequence of Eq(2.16) the coordinate y is an isotropic vector in the deformed 6D space

$$(y,y) = \varepsilon^{abcd}_q \, y_{ab} \, y_{cd} = 0 \tag{2.19}$$

This $GL_q(4,C)$ covariant equation determines the 4D deformed subspace of the complex 6D quantum plane. The classical analogue of this subspace is a complex sphere S^C_4.

Consider a duality transformation $*$ of the basic q-twistor 2-forms by analogy with Ref[7]

$$* \, dz \, dz' = dz \, dz' \, P_4^{(+)} - dz \, dz' \, P_4^{(-)} \tag{2.20}$$

where $P_4^{(\pm)}$ are the $GL_q(4)$ projectors (2.11). Note that a self-dual part $dz \, dz' \, P_4^{(+)}$ is proportional to the $SL_q(2)$-invariant conformal tensor

$$\varepsilon_{\alpha\beta}(q) \, dz_a^\alpha \, dz_b^\beta \tag{2.21}$$

Real q-twistors can be treated as a representation of the real quantum group $SL_q(2, R) \times GL_q(4, R)$. The classical analogue of these twistors are connected with the real pseudo-Euclidean $(2, 2)$-space [13]. Consider the R-matrices (2.3,2.5) and the conditions $|q| = 1$ and $|r(ab)| = 1$, then under the complex conjugation

$$\overline{R^{ab}_{cd}} = (R^{-1})^{ba}_{dc} \tag{2.22}$$

$$\overline{R^{\alpha\beta}_{\mu\nu}} = (R^{-1})^{\beta\alpha}_{\nu\mu} \tag{2.23}$$

These formulas correspond to anti-involution of the real $T_q(2, 2)$ twistors

$$\overline{z} = z, \qquad \overline{dz} = dz \tag{2.24}$$

$$\overline{z \, z'} = z' \, z, \qquad \overline{z \, dz'} = dz' \, z \tag{2.25}$$

The pseudoreal Euclidean q-twistors have a more complicated anti-involution

$$\overline{z_a^\alpha} = \varepsilon_{\alpha\beta}(q) z_b^\beta \, C_a^b(q) \tag{2.26}$$

where $C(q)$ is the charge-conjugation matrix for the Euclidean conformal quantum group $U_q^*(4) = D \times SU_q^*(4)$ and D is a real one-parameter dilatation.

It is convenient to use the simple representation

$$C(q) = \begin{pmatrix} \varepsilon^{\alpha\beta}(q) & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}}(q) \end{pmatrix} \tag{2.27}$$

For the real q we have the following properties:

$$\overline{\varepsilon^{\alpha\beta}(q)} = \varepsilon^{\alpha\beta}(q) = -\varepsilon_{\alpha\beta}(q) \tag{2.28}$$

$$\overline{C(q)} = C(q), \qquad C^2(q) = -I \tag{2.29}$$

$$\overline{\overline{z_a^\alpha}} = -\varepsilon^{\alpha\beta}(q) \varepsilon_{\beta\gamma}(q) z_b^\gamma \, (C^2)_a^b(q) = z_a^\alpha \tag{2.30}$$

$$\overline{R^{dc}_{ba}} = C_e^d(q) \, C_f^c(q) \, R^{fe}_{gh} \, C_a^g(q) \, C_b^h(q) \tag{2.31}$$

We do not here consider the conformal quantum group $SU_q(2, 2)$ [24,25] and corresponding q-twistors.

3 Quantum-group gauge theory on the q-twistor space

The classical gauge field on some domain $\{x^m\}$ of the basic space corresponds to the connection 1-form

$$A(x,dx) = dx^m A_m(x) \tag{3.1}$$

which can be decomposed in terms of the gauge-group generators. For the domain with the coordinates $\tilde{x}(x)$ one should define the transformed connection

$$\tilde{A}(\tilde{x},d\tilde{x}) = T(x)AT^{-1}(x) + dT(x)T^{-1}(x) \tag{3.2}$$

where $T(x)$ is a matrix of the local gauge transformation.

The components of the matrix $A(x,dx)$ satisfy the anticommutativity conditions

$$\{A^i_k, A^l_m\} = 0 \tag{3.3}$$

The classical gauge group formally has an infinite number of generators. A constructive example of the classical gauge algebra is the affine (Kac-Moody) algebra. The quantum affine algebras can be considered as a basis of the quantum gauge theory on the classical two-dimensional space.

The formal quantization of the gauge groups on the multi-dimensional classical or quantum spaces is a difficult problem. Let R_N be the constant R-matrix for the quantum group $GL_q(N)$ and x^M are the coordinates of some basic space. Consider the simplest possible relations for the components of the quantum gauge matrix

$$R_N T(x)T'(x) = T(x)T'(x)R_N \tag{3.4}$$

$$T^i_k(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(T^i_k)_{M_1\cdots M_n} x^{M_1}\ldots x^{M_n} \tag{3.5}$$

where $i,k\ldots = 1\ldots N$.

The quantum-group gauge matrix is well defined if the relation (3.4) generates the consistent set of relations between the coefficients $(T^i_k)_{M_1\cdots M_n}$.

Quantum deformations of the $GL(N)$ gauge connection can be treated in terms of the noncommutative gauge algebra for the components of the deformed connection 1-form [5,6]

$$\begin{gathered}(AR_NA + R_NAR_NAR_N)^{ik}_{mn} = \\ A^i_l(R_N)^{lk}_{rn}A^r_m + (R_N)^{ik}_{jl}A^j_r(R_N)^{rl}_{st}A^t_p(R_N)^{pt}_{mn} = 0\end{gathered} \tag{3.6}$$

These relations generalize the classical anticommutativity conditions (3.3). The gauge algebra is an analogue of the relations between components of the right-invariant 1-forms $\omega = dTT^{-1}$ in the framework of the bicovariant differential calculus on $GL_q(N)$ [20-23]

$$\omega R_N\omega + R_N\omega R_N\omega R_N = 0, \quad d\omega - \omega^2 = 0 \tag{3.7}$$

Thus, the form ω can be considered as a pure gauge $GL_q(N)$-field. The general $GL_q(N)$ connection A has the nontrivial curvature 2-form

$$F = dA - A^2 \tag{3.8}$$

Explicit constructions of the deformed gauge fields on the q-twistor space contain also the noncommutative elements (moduli) which generate some algebra B

$$A^i_k(z, dz, B) = dz^\alpha_a (A^a_\alpha)^i_k(z, B) \tag{3.9}$$

The appearance of additional noncommutative elements is necessary for the consistency of the algebra (3.6) and the relations of the q-twistor algebra (2.8,2.9).

The (anti)self-duality equation for the gauge field (3.9) can be defined with the help of the relations (2.20)

$$* F = \frac{1}{2}(*dz^\alpha_a dz^\beta_b) F^{ab}_{\alpha\beta}(z, B) = \pm F \tag{3.10}$$

or in terms of the deformed field-strength

$$[P_4^{(\pm)}]^{ab}_{cd} F^{cd}_{\alpha\beta}(z, B) = 0 \tag{3.11}$$

The solutions of this equation and the explicit construction of the algebra B will be considered in sections 4 and 5.

A quantum deformation of the Yang-Mills equation has the standard form in the framework of the external algebra $\Lambda T_q(4, C)$

$$\nabla * F = d * F + [A, *F] = 0 \tag{3.12}$$

The bicovariant differential calculus with the ordinary Leibniz rule for the operator d and the gauge-connection algebra (3.6) are consistent only for the case of the nonsemisimple quantum group $GL_q(N)$. The gauge algebra produce the restriction

$$\alpha = \mathrm{Tr}_q A \neq 0 \tag{3.13}$$

Nevertheless, one should use the gauge-covariant conditions [7]

$$d\alpha = 0, \quad \mathrm{Tr}_q A^2 = 0, \quad \mathrm{Tr}_q F = 0 \tag{3.14}$$

These restrictions generate the effective reduction of the Abelian gauge field α in the framework of the gauge group $GL_q(N)$.

4 Quantum deformations of the t'Hooft multi-instanton solution

A simple form of the manifest multi-instanton solution in the Euclidean space was discussed in Refs[26-28]. This solution can be written in terms of the potential Φ satisfying the Laplace equation. The classical twistor version of the t'Hooft solution was considered in Ref[13].

Consider firstly the deformed Laplace equation in the complex q-twistor space $T_q(4, C)$. Using Eq(2.13) one can obtain the action of the partial q-twistor derivative on the isotropic $6D$-vector

$$\partial^c_\alpha \, y_{ab} = \varepsilon_{\alpha\beta}(q) \, [P_4^{(-)}]^{dc}_{ba} \, z^\beta_d \tag{4.1}$$

Introduce the formal differential operator that acts only on the $6D$ vector variables

$$\partial^{dc} \triangleright y_{ab} = [P_4^{(-)}]^{dc}_{ba} \tag{4.2}$$

Now we can write the following relations:

$$dy_{ab} = \varepsilon_{\alpha\beta}(q)\, [P_4^{(-)}]^{dc}_{ba}\, dz^{\alpha}_{c}\, z^{\beta}_{d} \tag{4.3}$$
$$d\Phi(y) = dz^{\alpha}_{c}\, \partial^{c}_{\alpha}\, \Phi(y) = dy_{ab}\, \partial^{ba}\, \Phi(y) \tag{4.4}$$
$$\partial^{c}_{\alpha}\, \Phi(y) = \varepsilon_{\alpha\beta}(q)\, z^{\beta}_{b}\, \partial^{bc}\, \Phi(y) \tag{4.5}$$

The $SL_q(2,C)$-invariant analogue of the Laplace operator has the following form:

$$\Delta^{ba} = -\frac{q}{1+q^2}\varepsilon^{\alpha\beta}(q)\, \partial^{b}_{\beta}\, \partial^{a}_{\alpha} \tag{4.6}$$

By definition, we have

$$\Delta^{ba} \triangleright y_{cd} = [P_4^{(-)}]^{ba}_{dc} = \partial^{ba} \triangleright y_{cd} \tag{4.7}$$

In this section we shall use the standard $GL_q(4,C)$ R-matrix corresponding to Eq(2.3) with $r(ab) = 1$. This R-matrix satisfies the following identity:

$$\varepsilon_q^{abcd}\, R^{a'h}_{ea}\, R^{b'e}_{fb}\, R^{c'f}_{gc}\, R^{d'g}_{h'd} = q\, \delta^{h}_{h'}\, \varepsilon_q^{a'b'c'd'} \tag{4.8}$$

Introduce the additional noncommutative moduli b^p_{ab} where p is an arbitrary number

$$(b^p, b^p) = \varepsilon_q^{abcd} b^p_{ab}\, b^p_{cd} = 0 \tag{4.9}$$
$$y_{ab}\, b^p_{cd} = R^{ea'}_{ga}\, R^{fg}_{cb}\, R^{c'b'}_{he}\, R^{d'h}_{df}\, b^p_{a'b'}\, y_{c'd'} \tag{4.10}$$
$$b^p_{ab}\, b^{\hat{p}}_{cd} = q^{-2} R^{ea'}_{ga}\, R^{fg}_{cb}\, R^{c'b'}_{he}\, R^{d'h}_{df}\, b^{\hat{p}}_{a'b'}\, b^p_{c'd'} \tag{4.11}$$

and $p \le \hat{p}$ in the last equation.

This (B, y)-algebra has the following central elements

$$X_p = (y, b^p) = \varepsilon_q^{abcd}\, y_{ab}\, b^p_{cd} \tag{4.12}$$

The commutativity of X_p with y and $b^{\hat{p}}$ can be proved with the help of Eq(4.8).

Let us introduce the commutation relations between b^p, and z

$$b^p_{ab}\, z^{\gamma}_{c} = R^{eh}_{ga}\, R^{fg}_{cb}\, z^{\gamma}_{h}\, b^p_{ef} \tag{4.13}$$

An analogous relation for b^p, and dz can be obtained as the external derivative d of this formula by using $[d, b^p] = 0$.

Equation(2.9) generates the relation for y and dz

$$y_{ab}\, dz^{\gamma}_{c} = q\, R^{eh}_{ga}\, R^{fg}_{cb}\, dz^{\gamma}_{h}\, y_{ef} \tag{4.14}$$

Write the corresponding relation for the elements X_p

$$X_p\, dz^{\alpha}_{a} = q^2\, dz^{\alpha}_{a}\, X_p \tag{4.15}$$

Now one can determine the derivative of the central functions

$$\partial^a_\alpha \frac{1}{X_p} = -\frac{1}{q^2 X_p^2}\, \partial^a_\alpha X_p \tag{4.16}$$

$$\partial^a_\alpha \frac{1}{X_p^2} = -\frac{1+q^2}{q^4}\frac{1}{X_p^3}\partial^a_\alpha X_p \tag{4.17}$$

It is easy to check the following identity for the isotropic vectors b^p :

$$\varepsilon^{\alpha\beta}(q)\, \partial^b_\beta X_p \partial^a_\alpha X_p = -q^{-1}\varepsilon_q^{abcd}\, b^p_{cd}\, X_p \tag{4.18}$$

We can obtain the solutions of the deformed Laplace equation
(q-harmonic functions)

$$\Delta^{ba}\, \frac{1}{X_p} = \frac{1}{q^5(1+q^2)}\frac{1}{X_p^2}\varepsilon^{\alpha\beta}(q)\, [\partial^b_\beta\, \partial^a_\alpha X_p - (1+q^2)\frac{1}{X_p}\partial^b_\beta X_p \partial^a_\alpha X_p] = 0 \tag{4.19}$$

By analogy with Ref[13] one can consider the deformed t'Hooft Ansatz for the $GL_q(2)$ self-dual gauge field

$$A^\alpha_\beta = q^{-3}\, dz^\alpha_a\, (\partial^a_\mu \Phi)\, \Phi^{-1}\varepsilon^{\sigma\mu}(q)\varepsilon_{\sigma\beta}(q) \tag{4.20}$$

$$\mathrm{Tr}_q A = -q^3 d\Phi\, \Phi^{-1}, \quad \mathrm{Tr}_q dA = 0 \tag{4.21}$$

where the potential function Φ for the instanton number P is a sum of q-harmonic functions with different elements b^p

$$\Phi = \sum_{p=1}^{P} \frac{1}{X_p} = \sum_{p=1}^{P} (y, b^p)^{-1} \tag{4.22}$$

The anti-self-dual part of the corresponding curvature form vanishes in consequence of Eq(4.19)

$$(F - *F)^\alpha_\beta\ \sim\ dz^\alpha_e\, dz^\gamma_f\, [P_4^{(-)}]^{fe}_{ac}\, \Delta^{ac}\Phi\, \Phi^{-1}\, \varepsilon_{\beta\gamma}(q) = 0 \tag{4.23}$$

Note that the isotropic vector b^p has 5 independent elements so (4.20) is the $5P$-parameter solution.

5 Quantum deformations of the ADHM-solution

The covariant formulation of the ADHM multi-instanton solution in the classical twistor space was considered in Ref[13]. We shall discuss the quantum deformations of this formalism.

Let us consider the gauge group $GL_q(N, C)$. The ADHM-solution for the instanton number p can be connected with some $GL_q(N + 2p, C)$ matrix q-twistor function. Introduce the notation for indices of different types : $A, B \ldots = 1 \ldots p;\ \ I, K, L, M \ldots = 1 \ldots N{+}2p$ and $i, k, l \ldots = 1 \ldots N$. The ADHM Ansatz for the general self-dual $GL_q(N, C)$ field contains the deformed twistor functions $u^i_I(z)$ and $\tilde{u}^I_i(z)$

$$A^i_k = du^i_I\, \tilde{u}^I_k, \quad u^i_I\, \tilde{u}^I_k = \delta^i_k \tag{5.1}$$

The commutation relations for the u and $\tilde{u}$ twistors are

$$(R_N)^{ik}_{lm}\ u^l_I\ u^m_K = u^i_L\ u^k_M\ \mathbf{R}^{LM}_{IK} \tag{5.2}$$
$$\mathbf{R}^{KI}_{ML}\ \tilde{u}^L_i\ \tilde{u}^M_k = \tilde{u}^I_l\ \tilde{u}^K_m\ (R_N)^{ml}_{ki} \tag{5.3}$$
$$\tilde{u}^I_l\ (R_N)^{li}_{mk}\ u^m_K = u^i_L\ \mathbf{R}^{IL}_{KM}\ \tilde{u}^M_k \tag{5.4}$$

where the R-matrices for $GL_q(N,C)$ and $GL_q(N+2p,C)$ are used.

Introduce also the relation for the differentials du

$$\tilde{u}^I_i\ (R_N)^{ik}_{lm}\ du^l_K = du^k_L\ (\mathbf{R}^{-1})^{IL}_{KM}\ \tilde{u}^M_m \tag{5.5}$$
$$du^i_L\ du^k_M\ (\mathbf{R}^{-1})^{LM}_{IK} = -(R^{-1}_N)^{ik}_{lm}\ du^l_I\ du^m_K \tag{5.6}$$

These relations are necessary for proving a validity of the gauge algebra (3.6) in the framework of the ADHM-Ansatz

$$(AR_NA)^{ik}_{mn} = du^i_I\ du^k_L\ (\mathbf{R}^{-1})^{IL}_{KM}\ \tilde{u}^M_n\ \tilde{u}^K_m = -(R_NAR_NAR_N)^{ik}_{mn} \tag{5.7}$$

Consider also the linear twistor functions v and $\tilde{v}$

$$v^{A\alpha}_I = z^\alpha_a\ b^{aA}_I \tag{5.8}$$
$$\tilde{v}^{IA\alpha} = z^\alpha_a\ \tilde{b}^{aIA} \tag{5.9}$$

where b and $\tilde{b}$ are the noncommutative q-instanton moduli

$$b^{aA}_I\ z^\alpha_b = R^{da}_{cb}\ z^\alpha_d\ b^{cA}_I \tag{5.10}$$
$$\tilde{b}^{aIA}\ z^\alpha_b = R^{da}_{cb}\ z^\alpha_d\ \tilde{b}^{cIA} \tag{5.11}$$

The relations between b and $\tilde{b}$ will be defined below.

Introduce the following condition for the functions v and $\tilde{v}$:

$$v^{A\alpha}_I\ \tilde{v}^{IB\beta} = g^{AB}(z)\ \varepsilon^{\alpha\beta}(q) \tag{5.12}$$

where $g(z)$ is the nondegenerate $(p\times p)$ matrix with the central elements

$$g^{AB}(z) = q^{-2}\ y_{cd}\ b^{cA}_I\ \tilde{b}^{dIB} \tag{5.13}$$

The condition (5.12) is equivalent to the restriction on the elements of the B-algebra

$$[P^{(+)}]^{ab}_{cd}\ b^{cA}_I\ \tilde{b}^{dIB} = 0 \tag{5.14}$$

Write the basic commutation relations of the B-algebra

$$R^{ab}_{cd}\ b^{cA}_I\ b^{dB}_K = b^{aB}_L\ b^{bA}_M\ \mathbf{R}^{ML}_{KI} \tag{5.15}$$
$$\mathbf{R}^{IK}_{LM}\ \tilde{b}^{aLA}\ \tilde{b}^{bMB} = R^{ab}_{cd}\ \tilde{b}^{cIB}\ \tilde{b}^{dKA} \tag{5.16}$$
$$R^{ab}_{cd}\ b^{cA}_I\ \tilde{b}^{dKB} = \mathbf{R}^{KL}_{IM}\ \tilde{b}^{aMB}\ b^{bB}_L \tag{5.17}$$

Remark that a formal permutation of the indices A and B is commutative.

Consider the new functions

$$\tilde{v}^I_{A\alpha} = \tilde{v}^{IB\beta}\ g_{BA}(z)\ \varepsilon_{\beta\alpha}(q) \tag{5.18}$$

where we use the matrix g_{BA} inverse of the matrix (5.13)

$$g_{BA}(z)\ g^{AC}(z) = \delta^C_B \tag{5.19}$$

Now one can construct the full quantum $GL_q(N+2p,C)$ matrices

$$\mathbf{U} = \begin{pmatrix} u^i_I \\ v^{A\alpha}_I \end{pmatrix}, \qquad \mathbf{S}(\mathbf{U}) = \mathbf{U}^{-1} = \begin{pmatrix} \tilde{u}^I_i \\ \tilde{v}^I_{A\alpha} \end{pmatrix} \tag{5.20}$$

The standard $GL_q(N+2p,C)$ commutation relations for these matrices contain Eqs(5.2-5.4) and the relations for v and $\tilde{v}$ functions

$$\widetilde{\mathbf{R}}\ \mathbf{U}\ \mathbf{U}' = \mathbf{U}\ \mathbf{U}'\ \mathbf{R} \tag{5.21}$$

$$\mathbf{R}\ \mathbf{S}'\ \mathbf{S} = \mathbf{S}'\ \mathbf{S}\ \widetilde{\mathbf{R}} \tag{5.22}$$

$$\mathbf{S}\ \widetilde{\mathbf{R}}\ \mathbf{U} = \mathbf{U}'\ \mathbf{R}\ \mathbf{S}' \tag{5.23}$$

where the R-matrix for $GL_q(N+2p,C)$ can be written in the following form

$$\widetilde{\mathbf{R}} = \begin{pmatrix} (R_N)^{ik}_{mn} & 0 & 0 & 0 \\ 0 & \lambda\delta^A_C\delta^\alpha_\mu\delta^k_n & \delta^A_D\delta^\alpha_\nu\delta^k_m & 0 \\ 0 & \delta^i_n\delta^B_C\delta^\beta_\mu & 0 & 0 \\ 0 & 0 & 0 & \delta^A_D\delta^B_C R^{\alpha\beta}_{\mu\nu} \end{pmatrix}$$

where $\lambda = q - q^{-1}$.

The equations (5.5,5.6) follow from are the relations

$$\mathbf{S}\ \widetilde{\mathbf{R}}\ d\mathbf{U} = d\mathbf{U}'\ \mathbf{R}^{-1}\ \mathbf{S}' \tag{5.24}$$

It should be stressed that the bicovariant differential calculus on $GL_q(N+2p,C)$ is the basis of the deformed ADHM-construction for the group $GL_q(N,C)$.

Write explicitly the orthogonality and completeness conditions for the deformed ADHM-twistors:

$$u^i_I\ \tilde{v}^{IA\alpha} = 0 \tag{5.25}$$

$$v^{A\alpha}_I\ \tilde{u}^I_i = 0 \tag{5.26}$$

$$\delta^I_K = \tilde{u}^I_i\ u^i_K\ +\ \tilde{v}^{IA\alpha}\ g_{AB}(z)\varepsilon_{\alpha\beta}(q)\ v^{B\beta}_K \tag{5.27}$$

Now we are in a position to verify the self-duality of the connection (5.1)

$$\begin{aligned} dA^i_k - A^i_l\ A^l_k = du^i_I\ (\tilde{u}^I_l\ u^l_M\ -\ \delta^I_M)d\tilde{u}^M_k = \\ = -q^{-4}u^i_I\ \tilde{b}^{cIA}D^a_c\ g_{AB}(z)\varepsilon_{\alpha\beta}(q)\ dz^\alpha_a\ dz^\beta_b\ b^{bB}_M\ \tilde{u}^M_k \end{aligned} \tag{5.28}$$

where D^a_c is the $GL_q(4)$ metric. This curvature contains only the self-dual q-twistor 2-form (2.21).

The real forms of the deformed ADHM-construction are based on the quantum groups $U_q(N)$ and $GL_q(N,R)$.

It should be stressed that all R-matrices of our deformation scheme satisfy the Hecke relation with the common parameter q. The other possible parameters of different R-matrices are independent. The case $q=1$ corresponds to the unitary deformations ($R^2 = I$) of the twistor space and the gauge groups. It is evident that the trivial deformation of the z-twistors is consistent with the nontrivial unitary deformation of the gauge sector and vice versa.

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